

UNIQUENESS OF SOLUTIONS TO THE 3D QUINTIC GROSS-PITAEVSKII HIERARCHY

YOUNGHUN HONG, KENNETH TALIAFERRO, AND ZHIHUI XIE

ABSTRACT. In this paper, we study solutions to the three-dimensional quintic Gross-Pitaevskii hierarchy. We prove unconditional uniqueness among all small solutions in the critical space \mathfrak{H}^1 (which corresponds to H^1 on the NLS level). With slight modifications to the proof, we also prove unconditional uniqueness of solutions to the Hartree hierarchy without a smallness condition. Our proof uses the quantum de Finetti theorem, and is an extension of the work by Chen-Hainzl-Pavlović-Seiringer [4], and our previous work [15].

1. INTRODUCTION

1.1. Statement of the main result. In this paper, we establish uniqueness of small solutions to the three-dimensional quintic Gross-Pitaevskii (GP) hierarchy in the scaling-critical Sobolev type space.

The 3d quintic GP hierarchy is an infinite system of coupled linear equations

$$i\partial_t \gamma^{(k)} = (-\Delta_{\underline{x}_k} + \Delta_{\underline{x}'_k}) \gamma^{(k)} + \lambda \sum_{j=1}^k B_{j;k+1,k+2} \gamma^{(k+2)}, \quad k \in \mathbb{N}, \quad (1.1)$$

where $\gamma^{(k)} = \gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) : [0, T) \times \mathbb{R}^{3k} \times \mathbb{R}^{3k} \rightarrow \mathbb{C}$, the underlined variables \underline{x}_k and \underline{x}'_k denote k -tuples of spacial variables, i.e., $\underline{x}_k = (x_1, x_2, \dots, x_k) \in \mathbb{R}^{3k}$ and $\underline{x}'_k = (x'_1, x'_2, \dots, x'_k) \in \mathbb{R}^{3k}$, and the Laplacians are given by $\Delta_{\underline{x}_k} := \sum_{j=1}^k \Delta_{x_j}$ and $\Delta_{\underline{x}'_k} := \sum_{j=1}^k \Delta_{x'_j}$. We assume that for each $k \in \mathbb{N}$, $\gamma^{(k)}$ is a symmetric marginal density matrix such that

$$\gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) = \overline{\gamma^{(k)}(t, \underline{x}'_k; \underline{x}_k)} \quad (1.2)$$

and

$$\gamma^{(k)}(t, x_{\sigma(1)}, \dots, x_{\sigma(k)}; x'_{\sigma'(1)}, \dots, x'_{\sigma'(k)}) = \gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) \quad (1.3)$$

for any permutations σ and σ' on $\{1, 2, \dots, k\}$. The *contraction operator* $B_{j;k+1,k+2}$ is defined by

$$\begin{aligned} & B_{j;k+1,k+2} \gamma^{(k+2)}(t, \underline{x}_k; \underline{x}'_k) \\ &:= \int dx_{k+1} dx_{k+2} dx'_{k+1} dx'_{k+2} [\delta(x_j - x_{k+1}) \delta(x_j - x_{k+2}) \delta(x_j - x'_{k+1}) \delta(x_j - x'_{k+2}) \\ &\quad - \delta(x'_j - x_{k+1}) \delta(x'_j - x_{k+2}) \delta(x'_j - x'_{k+1}) \delta(x'_j - x'_{k+2})] \gamma^{(k+2)}(t, \underline{x}_{k+2}; \underline{x}'_{k+2}) \\ &= \gamma^{(k+2)}(t, \underline{x}_k, x_j, x_j; \underline{x}'_k, x_j, x_j) - \gamma^{(k+2)}(t, \underline{x}_k, x'_j, x'_j; \underline{x}'_k, x'_j, x'_j). \end{aligned} \quad (1.4)$$

The coupling constant is either -1 or 1 . We call the GP hierarchy (1.1) *defocusing* if $\lambda = 1$, and *focusing* if $\lambda = -1$.

To define solutions to the GP hierarchy, we introduce the following definitions (see also [4, 9, 10, 11, 12]). For $s \geq 0$, we define the homogeneous Sobolev space $\dot{\mathfrak{H}}^s$ for sequences by

$$\dot{\mathfrak{H}}^s := \left\{ \{\gamma^{(k)}\}_{k \in \mathbb{N}} : \text{Tr}(|R^{(k,s)} \gamma^{(k)}|) < M^{2k} \text{ for some positive constant } M < \infty \right\} \quad (1.5)$$

where

$$R^{(k,s)} := \prod_{j=1}^k (-\Delta_{x_j})^{\frac{s}{2}} (-\Delta_{x'_j})^{\frac{s}{2}}.$$

Similarly, we define the inhomogeneous Sobolev space \mathfrak{H}^s for sequences by

$$\mathfrak{H}^s := \left\{ \{\gamma^{(k)}\}_{k \in \mathbb{N}} : \text{Tr}(|S^{(k,s)}\gamma^{(k)}|) < M^{2k} \text{ for some constant } M < \infty \right\} \quad (1.6)$$

where

$$S^{(k,s)} := \prod_{j=1}^k (1 - \Delta_{x_j})^{\frac{s}{2}} (1 - \Delta_{x'_j})^{\frac{s}{2}}.$$

A sequence $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$ is called a *mild solution* in $L_{t \in [0,T]}^\infty \mathfrak{H}^s$ (or $L_{t \in [0,T]}^\infty \mathfrak{H}^s$) to the quintic GP hierarchy if it solves the hierarchy of the integral equations

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\lambda \sum_{j=1}^k \int_0^t U^{(k)}(t-s) B_{j;k+1,k+2} \gamma^{(k+2)}(s) ds, \quad \forall k \in \mathbb{N}, \quad (1.7)$$

where $U^{(k)}(t) := e^{it(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})}$ is the free evolution operator. A sequence $\{\gamma^{(k)}\}_{k \in \mathbb{N}}$ is called *admissible* if for each $k \in \mathbb{N}$ and $t \in [0, T]$, $\gamma^{(k)}$ is a non-negative trace class operator on $L_{sym}^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})$ (subset of L^2 functions that satisfy (1.3)) and

$$\gamma^{(k)} = \text{Tr}_{k+1}(\gamma^{(k+1)}) = \int_{\mathbb{R}^3} dx_{k+1} \gamma^{(k+1)}(\underline{x}_k, x_{k+1}; \underline{x}'_k, x_{k+1}). \quad (1.8)$$

We call a sequence $\{\gamma^{(k)}\}_{k \in \mathbb{N}}$ a *limiting hierarchy* if there is a sequence $\{\gamma_N^{(N)}\}_{N \in \mathbb{N}}$ of non-negative density matrices on $L_{sym}^2(\mathbb{R}^{3N} \times \mathbb{R}^{3N})$ with $\text{Tr}(\gamma_N^{(N)}) = 1$ such that $\gamma^{(k)}$ is the weak-* limit of the k -particle marginals of $\gamma_N^{(N)}$ in the trace class on $L_{sym}^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})$, that is,

$$\begin{aligned} \gamma_N^{(k)} &:= \text{Tr}_{k+1, \dots, N}(\gamma_N^{(N)}) \\ &= \int_{\mathbb{R}^{3(N-k)}} dx_{k+1} \cdots dx_N \gamma_N^{(N)}(\underline{x}_k, x_{k+1}, \dots, x_N; \underline{x}'_k, x_{k+1}, \dots, x_N) \\ &\rightharpoonup^* \gamma^{(k)} \text{ as } N \rightarrow \infty. \end{aligned} \quad (1.9)$$

In this paper, we consider mild solutions to the GP hierarchy (1.1) that are admissible or limiting hierarchies. Such mild solutions are physically relevant in the theory of derivation of the nonlinear Schrödinger equation (NLS) from the many body linear Schrödinger equation (see Section 1.2).

We now state our main result.

Theorem 1.1 (Uniqueness of small solutions to the quintic GP hierarchy). *Suppose that $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$ is a mild solution in $L_{t \in [0,T]}^\infty \mathfrak{H}^1$ to the quintic GP hierarchy (1.1) with initial data $\{\gamma^{(k)}(0)\}_{k \in \mathbb{N}}$, which is either admissible or a limiting hierarchy for each t . If $\text{Tr}(|R^{(k,1)}\gamma^{(k)}|) < M^{2k}$ for all $t \in [0, T]$ for $M > 0$ sufficiently small, then $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$ is the only such solution for the given initial data.*

The quintic GP hierarchy is closely related to the quintic NLS via factorized functions. Indeed, one can check that if ϕ_t is a solution to the quintic NLS

$$i\partial_t \phi_t = (-\Delta)\phi_t + \lambda|\phi_t|^4 \phi_t, \quad (1.10)$$

then a sequence of factorized functions,

$$\gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) = (|\phi_t\rangle\langle\phi_t|)^{\otimes k} := \prod_{j=1}^k \phi_t(x_j) \overline{\phi_t(x'_j)}, \quad (1.11)$$

solves the GP hierarchy (1.1). In this sense, proving uniqueness for the GP hierarchy is more difficult than it is for the quintic NLS.

The quintic GP hierarchy was studied by T. Chen and Pavlović [5] for the derivation of the quintic NLS as the Gross-Pitaevskii field limit of a non-relativistic Bose gas with 3-particle interactions. As a part of their analysis, the authors proved (conditional) uniqueness of solutions to the quintic GP hierarchy in an energy space, that is, a Sobolev type space of order 1, in one and two dimensions. We remark that in all dimensions, proving such uniqueness in an energy space is necessary to derive NLS. However, it is an open problem to prove uniqueness in three dimensions.

Theorem 1.1 provides an answer for this open problem under a smallness assumption. We remark that the 3d quintic GP hierarchy is scaling-critical in $\dot{\mathfrak{H}}^1$, and that even with our smallness assumption, our theorem is the first uniqueness theorem for the cubic or quintic GP hierarchy in a scaling-critical space. Moreover, uniqueness in Theorem 1.1 is unconditional.

It remains an open problem to remove the smallness assumption. In the case of the 3d quintic NLS, it is known that solutions are unique in the space H^s for $s \geq 1$, without a smallness assumption [3, 7, 13, 17]. However, the proof of unconditional uniqueness in the scaling-critical case $s = 1$ differs from the proof in the subcritical case $s > 1$. In the case of the 3d quintic GP hierarchy, we also expect that an approach different from the one that we use in the scaling-subcritical case is needed to remove the smallness assumption in the scaling-critical case. Currently, the main obstacle to removing the smallness assumption for solutions to the 3d quintic GP hierarchy in the scaling-critical case is the generally infinite cardinality of the support of the measure μ in the statement of the quantum de Finetti theorem, Theorem 2.1.

To compare scaling-critical and subcritical regimes, we provide a uniqueness theorem for the 3d quintic Hartree hierarchy. The 3d quintic Hartree hierarchy is also an infinite hierarchy as (1.1). However the contraction operator $B_{j,k+1,k+2}$ in (1.4) is replaced by

$$\begin{aligned} & B_{j,k+1,k+2} \gamma^{(k+2)}(t, \underline{x}_k; \underline{x}'_k) \\ & := \int dx_{k+1} dx_{k+2} dx'_{k+1} dx'_{k+2} \\ & \quad V(x_j - x_{k+1}, x_j - x_{k+2}) V(x_j - x'_{k+1}, x_j - x'_{k+2}) \gamma^{(k+2)}(t, \underline{x}_{k+2}; \underline{x}'_{k+2}) \\ & \quad - \int dx_{k+1} dx_{k+2} dx'_{k+1} dx'_{k+2} \\ & \quad \quad V(x'_j - x_{k+1}, x'_j - x_{k+2}) V(x'_j - x'_{k+1}, x'_j - x'_{k+2}) \gamma^{(k+2)}(t, \underline{x}_{k+2}; \underline{x}'_{k+2}). \end{aligned} \tag{1.12}$$

Note that the 3d quintic Hartree equation is subcritical in $L_{t \in [0, T]}^\infty \dot{\mathfrak{H}}^1$ if the three-particle interaction potential V is less singular than the product of delta functions. This is, if $V(\cdot, \cdot) \in L_{x,y}^r(\mathbb{R}^3 \times \mathbb{R}^3)$ for some $r > 1$. In this case, we can show unconditional uniqueness for the 3d quintic Hartree hierarchy without a smallness assumption.

Theorem 1.2 (Unconditional uniqueness for the quintic Hartree hierarchy). *Suppose that $V(\cdot, \cdot) \in L_{x,y}^r(\mathbb{R}^3 \times \mathbb{R}^3)$ for some $r > 1$. Let $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}} \in \dot{\mathfrak{H}}^1$ be a mild solution to the quintic Hartree hierarchy (1.7) with initial data $\{\gamma^{(k)}(0)\}_{k \in \mathbb{N}}$, which is either admissible or a limiting hierarchy for each t . If there exists $M > 0$ such that $\text{Tr}(|R^{(k,1)} \gamma^{(k)}|) < M^{2k}$ for all $t \in [0, T]$, then $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$ is the only such solution for the given initial data.*

1.2. Related works. The background work in this line goes back to the derivation of Schrödinger type equations from interacting particle systems. In the pioneering works by Hepp [14], Spohn [22] and in a series of more recent breakthroughs by Erdős, Schlein and Yau [9, 10, 11, 12], the authors derived the cubic NLS in \mathbb{R}^3 . A major ingredient in this derivation is the establishment the uniqueness of solutions to the corresponding GP hierarchy. The proof of uniqueness by Erdős-Schlein-Yau requires sophisticated Feynman graph expansions. Later, Klainerman and Machedon [19] rephrased

this as a board game argument to provide an alternative approach to prove uniqueness of solutions. However, the result in [19] is conditional in that the solutions that satisfy an a-priori space-time bound assumption. This assumption is used by Kirkpatrick, Schlein, and Staffilani [18] in two dimensional settings for compact and non-compact domains.

A recent new proof on the unconditional uniqueness of 3d cubic GP hierarchy was initiated by T.Chen, Hainzl, Pavlović and Seringer [4] using the *quantum de Finetti theorem*. The quantum de Finetti theorem is a quantum analogue of the Hewitt-Savage theorem in probability theory. The strong version of the quantum de Finetti theorem (see 2.1) asserts that an infinite sequence of *admissible* marginal density matrices can be expressed as an average over factorized states. However, for each t , the limiting hierarchies of density matrices do not necessarily satisfy admissibility. In this case, one uses the weak version of the de Finetti theorem (see 2.2). This is necessary when working with the BBGKY hierarchy approach for the derivation of NLS as in [9, 10, 11, 12], where one starts with a finite BBGKY hierarchy of N equations for the bosonic N -particle system (see (2.1) in [10]). In this case, the GP hierarchy of equations is obtained by taking $N \rightarrow \infty$ in the finite hierarchy. As part of the derivation, one proves that the weak-* limit of solutions $\gamma_N^{(k)}$ to the BBGKY hierarchy solve the infinite GP hierarchy.

By taking advantage of the quantum de Finetti theorems that give an alternative factorized formula for the solutions to the hierarchy, the authors of [15] established unconditional uniqueness for cubic GP hierarchy at the same regularity level of the corresponding NLS. Others have also used the de Finetti theorem to prove unconditional uniqueness for GP hierarchies in various settings. In [21], V. Sohinger adapted the method from [4] to cubic GP hierarchy in a periodic setting. In [6], X.Chen-Smith studied a Chen-Simon-Schrödinger hierarchy.

1.3. Strategy of the proof. We prove Theorem 1.1 and Theorem 1.2 in the framework of Chen-Hainzl-Pavlović-Seringer [4]. Due to the linearity of the hierarchy, it suffices to show that solutions solution having a zero initial are the zero solution. In our proof, we iterate the Duhamel formula (1.7) with zero initial data n times, resulting in a number of terms that grows factorially in n . We reduce the number of terms by the Erdős-Schlein-Yau combinatorial argument in Klainerman-Machedon's formulation [19]. The quintic version of this combinatoric reduction was used by Chen-Pavlovic in [5]. We use it for the 3d quintic GP and Hartee hierarchies without modification. Next, we apply the quantum de Finetti theorem to write each term as an integral sum of factorized states, and reorganize them using a tree-graph structure (see Figure 1 below) which extends the tree-graph in Chen-Hainzl-Pavlović-Seiringer [4]. Then, we iteratively estimate the n integrals. In each step, we apply our multilinear estimates, which can be found in Appendix A. Finally, we send $n \rightarrow \infty$ and find that solutions having zero initial data must be the zero solution.

In our previous work [15], we proved unconditional uniqueness for the cubic GP hierarchy in a low regularity setting, using a similar approach. In [15], our key ingredients were the trilinear estimates (2.19), (2.21) and (2.23) in Lemma 2.6. These estimates are based on the dispersive estimates

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}(\mathbb{R}^d)}, \quad p \geq 2, \quad (1.13)$$

and negative order Sobolev norm estimates (Lemma A.3 in [15]). In the proof, we applied these estimates to the reorganized integrals iteratively together with multilinear estimates based on Strichartz estimates ((2.20), (2.22) and (2.24) in Lemma 2.6). We remark that the use of dispersive estimates is crucial in obtaining the optimal subcritical low regularity uniqueness theorem. The dispersive estimates don't work in the scaling-critical space, however. Roughly speaking, this is due to the failure of integrability (in time) of the bound in (1.13). For instance, if one tries to prove uniqueness for the 3d quintic GP hierarchy in $L_{t \in [0, T]}^\infty \mathfrak{H}^1$ by the same approach, one should choose $p = 6$ for the multilinear estimate. Then, the bound in (1.13) is not integrable in time.

In the present work, instead of using dispersive estimates, we use multilinear estimates (Proposition A.1 and Propositions A.3) that are based on by Strichartz estimates and a negative order Sobolev norm bound. In the case of the Hartree hierarchy, we also make use of a convolution estimates of W. Beckner [2].

1.4. Notation. In order to prove Theorem 1.1 and Theorem 1.2 at the same time, we define

$$V_\infty(y, z) := \begin{cases} V(y, z), & \text{for the Hartree hierarchy.} \\ \lambda \delta(y) \delta(z), & \text{for the GP hierarchy.} \end{cases} \quad (1.14)$$

With this notation, we can now combine definitions (1.4) and (1.12) of $B_{j;k+1,k+2}$ for the GP hierarchy and the Hartree hierarchy, respectively, as follows.

$$\begin{aligned} & B_{j;k+1,k+2} \gamma^{(k+2)}(t, \underline{x}_k; \underline{x}'_k) \\ &:= \int dx_{k+1} dx_{k+2} dx'_{k+1} dx'_{k+2} \\ & \quad V_\infty(x_j - x_{k+1}, x_j - x_{k+2}) V_\infty(x_j - x'_{k+1}, x_j - x'_{k+2}) \gamma^{(k+2)}(t, \underline{x}_{k+2}; \underline{x}'_{k+2}) \\ & \quad - \int dx_{k+1} dx_{k+2} dx'_{k+1} dx'_{k+2} \\ & \quad V_\infty(x'_j - x_{k+1}, x'_j - x_{k+2}) V_\infty(x'_j - x'_{k+1}, x'_j - x'_{k+2}) \gamma^{(k+2)}(t, \underline{x}_{k+2}; \underline{x}'_{k+2}). \end{aligned} \quad (1.15)$$

1.5. Organization of the paper. This paper is organized as follows. In section 2 we present the road map for the proof of the main theorems and reduce the the main theorems to Proposition 2.1. We illustrate with an example how to factorize solutions in section 3. In section 4, we introduce tree graphs to illustrate our decomposition of each factor, and present properties of the associated kernels. The proof of Proposition 2.1 occupies section 5. In appendix A, we prove several multilinear estimates that we use section 5.

2. OUTLINE OF THE PROOF

We describe the strategy to prove uniqueness in more detail.

2.1. Setup. Let $\{\gamma_1^{(k)}(t)\}_{k \in \mathbb{N}}$ and $\{\gamma_2^{(k)}(t)\}_{k \in \mathbb{N}}$ be two mild solutions in $L^\infty_{t \in [0, T)} \dot{\mathfrak{H}}^1$ that solve (1.7) with the same initial data, and are either admissible or limiting hierarchies. To prove uniqueness, we will show that their difference $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$, given by

$$\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t), \quad k \in \mathbb{N}, \quad (2.1)$$

is zero. By linearity, the difference $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$ solves the GP (or Hartree) hierarchy with zero initial data. Therefore, it suffices to prove the following.

Proposition 2.1. *Suppose that $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$ is a mild solution to (1.1) with zero initial data, and that it is either admissible or a limiting hierarchy.*

(i) If $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$ solves the quintic GP hierarchy and $\|\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}\|_{L^\infty_{t \in [0, T)} \dot{\mathfrak{H}}^1}$ is sufficiently small, then

$$\text{Tr}(|R^{(k, -1)} \gamma^{(k)}(t)|) = 0, \quad \forall k \in \mathbb{N}. \quad (2.2)$$

(ii) If $\{\gamma^{(k)}(t)\}_{k \in \mathbb{N}}$ solves the quintic Hartree hierarchy and $V \in L^{1+}$, then (2.2) holds.

2.2. Duhamel expansion. To show (2.2), we first generate a Duhamel expansion as follows. For each $k \in \mathbb{N}$, $\gamma^{(k)}(t)$ solves

$$\gamma^{(k)}(t) = i\lambda \sum_{j=1}^k \int_0^t U^{(k)}(t-t_1) B_{j;k+1,k+2} \gamma^{(k+2)}(t_1) dt_1. \quad (2.3)$$

Fix $k \in \mathbb{N}$. Iterating the integral equation (2.3) $(n-1)$ times, we write

$$\gamma^{(k)}(t) = (i\lambda)^n \int_{t_n \leq \dots \leq t_1 \leq t} U^{(k)}(t-t_1) B_{k+2} \dots U^{(k+2n-2)}(t_{n-1}-t_n) B_{k+2n} \gamma^{(k+2n)}(t_n) dt_1 \dots dt_n. \quad (2.4)$$

Here, for each $r \geq 1$, the *combined contraction operator* is the sum of $k+2(r-1)$ many operators,

$$B_{k+2r} := \sum_{j=1}^{k+2(r-1)} B_{j;k+2r-1,k+2r}.$$

For notational convenience, we introduce the following notation.

$$\begin{aligned} U_{j,j'}^{(i)} &:= U^{(i)}(t_j - t_{j'}), \\ \underline{t}_n &:= (t, t_1, \dots, t_n), \quad t_0 = t, \\ J^k(\underline{t}_n) &:= U_{0,1}^{(k)} B_{k+2} U_{1,2}^{(k+2)} B_{k+4} \dots U_{n-1,n}^{(k+2n-2)} B_{k+2n} \gamma^{(k+2n)}(t_n). \end{aligned}$$

Then $\gamma^{(k)}(t)$ in (2.4) can be expressed in a compact form as

$$\gamma^{(k)}(t) = (i\lambda)^n \int_{t_n \leq \dots \leq t_1 \leq t} J^k(\underline{t}_n) d\underline{t}_n. \quad (2.5)$$

One may have observed that for fixed k , the number of terms in $J^k(\underline{t}_n)$ is $k(k+2) \dots (k+2n-2) \sim \mathcal{O}((2n)!)$. This factorial growth on the number of Duhamel expansion terms is the first difficulty before we proceed with the proof of proposition 2.1. As a preparation, we will present a summary of the combinatorial reduction process in section 2.3 to reduce $J^k(\underline{t}_n)$ into a smaller number of terms that we can control.

2.3. Combinatorial reduction. In the celebrated works [9, 10, 11, 12], Erdős-Schlein-Yau developed a sophisticated combinatorial arguments to reduce the number of Duhamel terms. Later, Klainerman and Machedon [19] rephrased this as a board game, which was extended to the quintic GP hierarchy by Chen-Pavlović in [5]. Since we will use the same arguments, we only present the notation and key reduction steps in this section. We refer the readers to [5] for the proofs of the related lemmas and theorems.

Let σ be a map from $\{k+1, k+2, \dots, k+2n-1\}$ to $\{1, 2, 3, \dots, k+2n-2\}$ such that $\sigma(2) = 1$ and $\sigma(j) < j$ for all j . $\mathcal{M}_{k,n}$ denotes the set of all such mappings. Then we have that

$$J^k(\underline{t}_n) = \sum_{\sigma \in \mathcal{M}_{k,n}} J^k(\underline{t}_n; \sigma), \quad (2.6)$$

where

$$J^k(\underline{t}_n; \sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1,k+2} U_{1,2}^{(k+2)} \dots U_{n-1,n}^{(k+2n-2)} B_{\sigma(k+2n-1);k+2n-1,k+2n} (\gamma^{(k+2n)}(t_n)) \quad (2.7)$$

is a basic term in $J^k(\underline{t}_n)$.

Next, for each $\sigma \in \mathcal{M}_{k,n}$ there is a $(k+2n-1) \times n$ matrix corresponding to it. This matrix can be reduced to a special upper echelon matrix that corresponds to σ_s via finite many so called *acceptable moves*. This transformation defines an equivalence relation among all the maps in $\mathcal{M}_{k,n}$. If σ and σ_s are equivalent, we denote this equivalence by $\sigma \sim \sigma_s$. From each equivalence classes,

we pick one map that corresponds to a special upper echelon matrix, denote it by σ_s . Theorem 7.4 in [5] confirms that there is a subset $D_{\sigma_s, t} \subset [0, t]^n$, such that

$$\sum_{\sigma \sim \sigma_s} \int_0^t \dots \int_0^{t_{n-1}} J^k(\underline{t}_n; \sigma) dt_1 \dots dt_n = \int_{D_{\sigma_s, t}} J^k(\underline{t}_n; \sigma_s) dt_1 \dots dt_n. \quad (2.8)$$

Hence we have a new formula for $\gamma^{(k)}(t)$

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k, n}^s} \int_{D_{\sigma, t}} J^k(\underline{t}_n; \sigma) dt_n, \quad (2.9)$$

where $\mathcal{M}_{k, n}^s$ is the union of all maps that correspond to special upper echelon matrices. By Lemma 7.3 of [5], $\#(\mathcal{M}_{k, n}^s) \leq 2^{k+3n-2}$.[‡]

2.4. Quantum de Finetti theorem. After decomposing $\gamma^{(k)}$ into a sum, we use the *quantum de Finetti* theorems to express each term in a factorized form. The quantum de Finetti theorem has a strong and weak version, and pertains to bosonic density matrices that are either admissible or obtained as a weak-* limit, respectively. We state both the strong and weak versions [20] below to be used in section 2.3.

Theorem 2.1 (Strong quantum de Finetti theorem). *If a sequence $\{\gamma^{(k)}\}_{k \in \mathbb{N}}$ of bosonic density matrices on $L_{sym}^2(\mathbb{R}^{3k})$ is admissible, then there exists a unique Borel probability measure μ , supported on the unit sphere $S \subset L^2(\mathbb{R}^3)$ and invariant under multiplication of $\phi \in L^2(\mathbb{R}^3)$ by complex numbers of modulus one, such that*

$$\gamma^{(k)} = \int d\mu(\phi) (|\phi\rangle\langle\phi|)^{\otimes k}, \quad k \in \mathbb{N}. \quad (2.10)$$

Theorem 2.2 (Weak quantum de Finetti theorem). *If a sequence $\{\gamma^{(k)}\}_{k \in \mathbb{N}}$ of bosonic density matrices on $L_{sym}^2(\mathbb{R}^{3k})$ is a limiting hierarchy, then there exists a unique Borel probability measure μ , supported on the unit ball $\mathcal{B} \subset L^2(\mathbb{R}^3)$ and invariant under multiplication of $\phi \in L^2(\mathbb{R}^3)$ by complex numbers of modulus one, such that (2.10) holds.*

There are different formulations of these theorems that are used in different settings. The formulation for density matrices was presented in a paper Lewin, Nam and Rougerie [20], and in a paper by Ammari and Nier [1]. For additional results related the de Finetti theorems, we refer the reader to Diaconis and Freedman [8], Hudson and Moody [16], and Stormer [23].

To make sure the de Finetti theorems are applicable, we note that if $\{\gamma_1^{(k)}\}_k$ and $\{\gamma_2^{(k)}\}_k$ are admissible, then so is $\{\gamma^{(k)}\}_k$. Similarly, if both $\{\gamma_1^{(k)}\}_k$ and $\{\gamma_2^{(k)}\}_k$ are obtained from a weak-* limit, then so is $\{\gamma^{(k)}\}_k$. Thus by Theorem 2.1 and Theorem 2.2, we obtain

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k, n}^s} \int_{D_{\sigma, t}} dt_n \int d\mu_{\underline{t}_n} J^k(\underline{t}_n; \sigma). \quad (2.11)$$

where

$$J^k(\underline{t}_n; \sigma) = U_{0,1}^{(k)} B_{\sigma(k+1); k+1, k+2} U_{1,2}^{(k+2)} \dots U_{n-1, n}^{(k+2n-2)} B_{\sigma(k+2n-1); k+2n-1, k+2n} (|\phi\rangle\langle\phi|)^{(k+2n)}. \quad (2.12)$$

We remark that $J^k(\underline{t}_n; \sigma) = J^k(\underline{t}_n; \sigma; \underline{x}_k; \underline{x}'_k)$ depends on $\underline{x}_k, \underline{x}'_k$. We omit the spatial variables for simplicity. We note that each factor in

$$(|\phi\rangle\langle\phi|)^{(k+2n)}(\underline{x}_{k+2n}; \underline{x}'_{k+2n}) = \prod_{i=1}^{k+2n} (|\phi\rangle\langle\phi|)(x_i; x'_i)$$

[‡]The multiplier 2^{k+3n-2} is affordable to us, since it can be absorbed in $(CT)^n$.

is a one-particle kernel, and that we can further decompose $J^k(\underline{t}_n; \sigma)$ as

$$J^k(t, t_1, \dots, t_n; \sigma; \underline{x}_k; \underline{x}'_k) = \prod_{j=1}^k J_j^1(t, t_{l_j,1}, \dots, t_{l_j,m_j}; \sigma_j; x_j; x'_j). \quad (2.13)$$

To better explain the reduction procedure, we present an example in section 3, and then go back to the general case in section 4.

3. EXAMPLE FACTORIZATION

Consider $k = 2, n = 4$, and ρ a permutation of $\{1, 2, \dots, n\}$. The map σ_s is represented by the following upper echelon matrix (each highlighted entry in a row is to the left of each highlighted entry in a lower row)

$$\begin{pmatrix} t_{\rho^{-1}(1)} & t_{\rho^{-1}(2)} & t_{\rho^{-1}(3)} & t_{\rho^{-1}(4)} \\ \mathbf{B}_{1;3,4} & B_{1;5,6} & B_{1;7,8} & B_{1;9,10} \\ B_{2;3,4} & \mathbf{B}_{2;5,6} & B_{2;7,8} & B_{2;9,10} \\ 0 & B_{3;5,6} & B_{3;7,8} & B_{3;9,10} \\ 0 & B_{4;5,6} & \mathbf{B}_{4;7,8} & \mathbf{B}_{4;9,10} \\ 0 & 0 & B_{5;7,8} & B_{5;9,10} \\ 0 & 0 & B_{6;7,8} & B_{6;9,10} \\ 0 & 0 & 0 & B_{7;9,10} \\ 0 & 0 & 0 & B_{8;9,10} \end{pmatrix} \quad (3.1)$$

Then, we have

$$J^2(\underline{t}_4; \sigma) = U_{0,1}^{(2)} B_{1;3,4} U_{1,2}^{(4)} B_{2;5,6} U_{2,3}^{(6)} B_{4;7,8} U_{3,4}^{(8)} B_{4;9,10}. \quad (3.2)$$

We will organize the terms in expansion of $J^2(\underline{t}_4; \sigma)$ into two one-particle density matrices by examining the effect of the contraction operators starting with the last one on the RHS of (3.2). We denote each factor in the last term $(|\phi\rangle\langle\phi|)^{\otimes 10}$ by u_i , ordered by increasing index i , so that $(|\phi\rangle\langle\phi|)^{\otimes 10} = \otimes_{i=1}^{10} u_i$.

First of all, in (3.2), the last interaction operator $B_{4;9,10}$ contracts the factor u_4, u_9 and u_{10} , and leaves all other factors unchanged.

$$B_{4;9,10}(\otimes_{i=1}^{10} u_i) = u_1 \otimes u_2 \otimes u_3 \otimes \Theta_4 \otimes u_5 \cdots \otimes u_8, \quad (3.3)$$

where

$$\Theta_4 := B_{1;2,3}(u_4 \otimes u_9 \otimes u_{10}).$$

The index α in Θ_α associates Θ_α to the α -th interaction operator from the left in (3.2). Since we only run the expansion to the n -th level, we have $1 \leq \alpha \leq n$. In this specific case, $n = 4$, and the 4th interaction operator is $B_{4;9,10}$.

Next, $B_{4;7,8}$ contracts $U_{3,4}^{(8)} \Theta_4, U_{3,4}^{(8)} u_7$ and $U_{3,4}^{(8)} u_8$.

$$B_{4;7,8} U_{3,4}^{(8)}((3.3)) = (U_{3,4}^{(3)}(u_1 \otimes u_2 \otimes u_3)) \otimes \Theta_3 \otimes (U_{3,4}^{(2)}(u_5 \otimes u_6)), \quad (3.4)$$

where

$$\Theta_3 := B_{1;2,3}((U_{3,4}^{(1)} \Theta_4) \otimes (U_{3,4}^{(1)} u_7) \otimes (U_{3,4}^{(1)} u_8)).$$

Then, by the semigroup property, $U_{2,3}^{(i)} U_{3,4}^{(i)} = U_{2,4}^{(i)}$. The operator $B_{2;5,6}$ contracts $U_{2,4}^{(1)} u_2, U_{2,4}^{(1)} u_5$ and $U_{2,4}^{(1)} u_6$, which correspond to the 2nd, 5th, and 6th factors in (3.4). The other factors are left invariant.

$$B_{2;5,6} U_{2,3}^{(6)}((3.4)) = (U_{2,4}^{(1)} u_1) \otimes \Theta_2 \otimes (U_{2,4}^{(1)} u_3) \otimes (U_{2,3}^{(1)} \Theta_3), \quad (3.5)$$

where

$$\Theta_2 = B_{1;2,3}(U_{2,4}^{(3)}(u_2 \otimes u_5 \otimes u_6)).$$

Finally, $B_{1;3,4}$ contracts $U_{1,4}^{(1)}u_1$, $U_{1,4}^{(1)}u_3$, and $U_{1,3}^{(1)}\Theta_3$ and leaves other factors unchanged.

$$B_{1;3,4}U_{1,2}^{(4)}((3.5)) = \Theta_1 \otimes (U_{1,2}^{(1)}\Theta_2), \quad (3.6)$$

where

$$\Theta_1 = B_{1;2,3}((U_{1,4}^{(1)}u_1) \otimes (U_{1,4}^{(1)}u_3) \otimes (U_{1,3}^{(1)}\Theta_3)).$$

Therefore, J^2 can be factorized as

$$J^2 = (U_{0,1}^{(1)}\Theta_1) \otimes (U_{0,2}^{(1)}\Theta_2) := J_1^1 \otimes J_2^1. \quad (3.7)$$

Now J^2 in (3.7) has two factors J_j^1 (note $j \leq k = 2$), which are 1-particle matrices. The reason we have such a decomposition is that $B_{\sigma_1(r);r,r+1}$ only affects three u_i each time, and as the contraction processes, all the u_i might be divided into different groups by the contraction connectivity.

For $j = 1$, after replacing back $u_i = |\phi\rangle\langle\phi|$, $i \leq k + 2n = 10$, we have

$$J_1^1 = U_{0,1}^{(1)}B_{1;2,3}U_{1,3}^{(2)}B_{3;4,5}U_{3,4}^{(3)}B_{3;6,7}(|\phi\rangle\langle\phi|)^{\otimes 7} \quad (3.8)$$

where we relabel the index in operators $B_{\sigma_1(r);r,r+1}$ such that the interaction operators in (3.8) correspond to $B_{1;3,4}$, $B_{4;7,8}$, $B_{4;9,10}$ respectively, and leave the connectivity structure among them unchanged. The labeling of function σ_1 (see the notation in (2.13)) takes values $\sigma_1(2) = 1$, $\sigma_1(4) = 3$, and $\sigma_1(6) = 3$.

For $j = 2$, we perform the relabeling in the same spirit find that

$$J_2^1 = U_{0,2}^{(1)}B_{1;2,3}U_{2,4}^{(3)}(|\phi\rangle\langle\phi|)^{\otimes 3}, \quad (3.9)$$

where $\sigma_2(2) = 1$.

We note that for any $\ell < \ell'$, the interaction operators $B_{\sigma(\ell);\ell,\ell+1}$ and $B_{\sigma(\ell');\ell',\ell'+1}$ in J^2 (which are highlighted in (3.1)) belong to the same factor J_j^1 if either $\sigma(\ell) = \sigma(\ell')$ or $\sigma(\ell') = \ell$. In such cases, we consider them as being *connected*. This connectivity structure is exactly the key point of the Duhamel terms that we want to illustrate using *tree graphs*. We include the detailed definitions and descriptions in section 4.

We further note that each σ_j can be viewed as the restriction of σ to J_j^1 . We call factors that have a free propagator applied to each ϕ (like J_2^1) *regular*, and factors that have the contractions of $(|\phi\rangle\langle\phi|)^{\otimes 3}$ without free propagator in between (like J_1^1) *distinguished*.

4. TREE GRAPHS FOR THE GENERAL CASE

4.1. The tree graphs. We begin by recalling that, from (2.12), J^k is given by

$$J^k(\underline{t}_n; \sigma) = U_{0,1}^{(k)}B_{\sigma(k+1);k+1,k+2}U_{1,2}^{(k+2)} \cdots U_{n-1,n}^{(k+2n-2)}B_{\sigma(k+2n-1);k+2n-1,k+2n}(|\phi\rangle\langle\phi|)^{\otimes(k+2n)}.$$

where

$$(|\phi\rangle\langle\phi|)^{\otimes(k+2n)}(\underline{x}_{k+2n}; \underline{x}'_{k+2n}) = \prod_{i=1}^{k+2n} (|\phi\rangle\langle\phi|)(x_i; x'_i)$$

is a product of one-particle kernels. Since the free evolution operators $U_{j,j'}^{(i)}$ and the contraction operators $B_{\sigma(r);r,r+1}$ preserve the product structure, it follows that we can also decompose

$$J^k(t, t_1, \dots, t_n; \sigma; \underline{x}_k; \underline{x}'_k) = \prod_{j=1}^k J_j^1(t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; \sigma_j; x_j; x'_j) \quad (4.1)$$

into a product of one-particle kernels J_j^1 . We associate to this decomposition k disjoint tree graphs $\tau_1, \tau_2, \dots, \tau_k$. These graphs appear as *skeleton graphs* in [9, 10, 11, 12]. As in [4, 15], we assign *root*, *internal*, and *leaf* vertices to each tree τ_j .

- A *root* vertex labeled as W_j , $j = 1, 2, \dots, k$, to represent $J_j^1(x_j; x'_j)$.
- An *internal* vertex labeled by v_ℓ , $\ell = 1, 2, \dots, n$, corresponding to $B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell}$ and attached to the time variable t_ℓ .
- A *leaf* vertex u_i , $i = 1, 2, \dots, k + 2n$, representing each factor $(|\phi\rangle\langle\phi|)(x_i; x'_i)$.

Next, we connect the vertices with *edges*, as described below.

- If v_ℓ is the smallest value of ℓ such that $\sigma(k + 2\ell - 1) = j$, then we connect v_ℓ to the root vertex W_j and write $W_j \sim v_\ell$ (or equivalently $W_j \sim B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell}$). If there is no internal vertex connected to a root vertex W_j , then we connect W_j to the leaf u_j , and write $W_j \sim u_j$.
- For any $1 < \ell \leq n$, if $\exists \ell' > \ell$ such that $\sigma(k + 2\ell - 1) = \sigma(k + 2\ell' - 1)$ or $\sigma(k + 2\ell' - 1) = k + 2\ell - 1$, then we connect v_ℓ and $v_{\ell'}$ and write $v_\ell \sim v_{\ell'}$ (or equivalently $B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell} \sim B_{\sigma(k+2\ell'-1);k+2\ell'-1,k+2\ell'}$). In this case, we call v_ℓ the *parent vertex* of $v_{\ell'}$, and $v_{\ell'}$ the *child vertex* of v_ℓ . We denote the three child vertices of v_ℓ by $v_{k_-(\ell)}$, $v_{k(\ell)}$ and $v_{k_+(\ell)}$, with $k_-(\ell) < k(\ell) < k_+(\ell)$.
- When there is no internal vertex with $\ell' > \ell$ and $k + 2\ell - 1 = \sigma(k + 2\ell' - 1)$, we connect v_ℓ to the leaf vertices $u_{k+2\ell-1}$, $u_{k+2\ell}$ and write $v_\ell \sim (u_{k+2\ell-1}, u_{k+2\ell})$ (or equivalently $B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell} \sim (u_{k+2\ell-1}, u_{k+2\ell})$).

We remark that it follows from the construction above that each root vertex has only one child vertex, and each internal vertex has exactly three child vertices (which can be either internal and leaf). We call the tree τ_j *distinguished* if $v_n \in \tau_j$, and *regular* if $v_n \notin \tau_j$. The three leaves connected to v_n are called *distinguished leaf vertices*, and all other leaves are called *regular leaf vertices*. Clearly, there are $k - 1$ regular trees and one distinguished tree in each tree graph.

A sample tree graph is given in Figure 1, for J^k as in (3.2). Each tree τ_j has root vertex W_j , for $j = 1, 2$. The leaf vertices $u_1, u_3, u_4, u_7, u_8, u_9, u_{10}$ and the internal vertices v_1, v_3, v_4 (or $B_{1;3,4}, B_{4;7,8}, B_{4;9,10}$) are distinguished. τ_1 is the distinguished tree, and is drawn with thick edges. Tree τ_2 with vertices W_2, v_2, u_2, u_5, u_6 is the regular tree, and is drawn with thin edges.

4.2. The distinguished one particle kernel J_j^1 . Let τ_j denote the distinguished tree graph. It has m_j internal vertices $(v_{\ell_{j,\alpha}})_{\alpha=1}^{m_j}$ and $2m_j + 1$ leaf vertices $(u_{j,i})_{i=1}^{2m_j+1}$. We enumerate the internal vertices with $\alpha \in \{1, \dots, m_j\}$ and the leaf vertices with $\alpha \in \{m_j + 1, \dots, 3m_j + 1\}$. To simplify notation, we refer to the vertex $v_{j,\alpha}$ by its label α . We observe that J_j^1 has the form

$$\begin{aligned} & J_j^1(t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; \sigma_j) \\ &= U^{(1)}(t - t_1) \cdots U^{(1)}(t_{\ell_{j,1}-1} - t_{\ell_{j,1}}) B_{\sigma_j(2);2,3} \cdots \\ & \quad \cdots B_{\sigma_j(2\alpha-2);2\alpha-2,2\alpha-1} U^{(2\alpha-1)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha-1}+1}) \cdots U^{(2\alpha-1)}(t_{\ell_{j,\alpha}-1} - t_{\ell_{j,\alpha}}) B_{\sigma_j(2\alpha);2\alpha,2\alpha+1} \cdots \\ & \quad \cdots U^{(2m_j-1)}(t_{\ell_{j,m_j}-1} - t_{\ell_{j,m_j}}) B_{\sigma_j(2m_j);2m_j,2m_j+1} (|\phi\rangle\langle\phi|)^{\otimes(2m_j+1)}. \end{aligned} \quad (4.2)$$

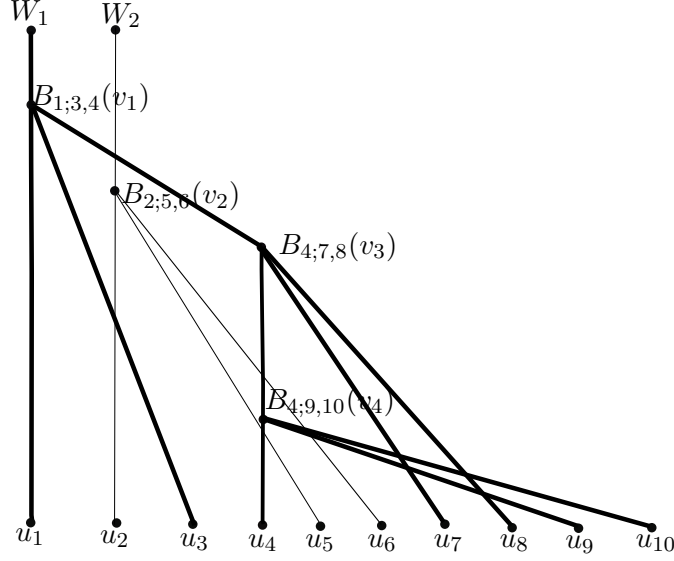


FIGURE 1. An example tree graph for J^k . It is a disjoint union of two trees τ_1 and τ_2 with root vertices W_1 and W_2 , respectively. Each tree corresponds to a one-particle kernel in the example in section 3, where $k = 2$ and $n = 4$.

By the semigroup property

$$U^{(\alpha)}(t)U^{(\alpha)}(s) = U^{(\alpha)}(t + s),$$

and the fact that $\sigma_j(2) = 1$, (4.2) reduces to

$$\begin{aligned} J_j^1(t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; \sigma_j) &= U^{(1)}(t - t_{\ell_{j,1}}) B_{1;2,3} \cdots \\ &\quad \cdots B_{\sigma_j(2\alpha-2); 2\alpha-2, 2\alpha-1} U^{(2\alpha-1)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha}}) B_{\sigma_j(2\alpha); 2\alpha, 2\alpha+1} \cdots \\ &\quad \cdots U^{(2m_j-1)}(t_{\ell_{j,m_j-1}} - t_{\ell_{j,m_j}}) B_{\sigma_j(2m_j); 2m_j, 2m_j+1} (|\phi\rangle\langle\phi|)^{\otimes(2m_j+1)}, \end{aligned} \quad (4.3)$$

where $\ell_{j,m_j} = n$.

4.3. Definition of the kernels Θ_α at the vertices of the distinguished tree graph. In this section, we proceed as in [4], and recursively assign a kernel Θ_α to each vertex α of the distinguished tree graph. The kernels at the vertices of the regular tree graph are defined similarly. We begin by assigning the kernel

$$\Theta_\alpha(x; x') := \phi(x)\bar{\phi}(x')$$

to the leaf vertex with label $\alpha \in \{m_j + 1, \dots, 3m_j + 1\}$.

Next, we determine Θ_{m_j} at the distinguished vertex $\alpha = m_j$ from the term on the last line of (4.3), given by

$$B_{\sigma_j(2m_j); 2m_j, 2m_j+1} (|\phi\rangle\langle\phi|)^{\otimes(2m_j+1)} = (|\phi\rangle\langle\phi|)^{\otimes(\sigma_j(2m_j)-1)} \otimes \Theta_{m_j} \otimes (|\phi\rangle\langle\phi|)^{\otimes(2m_j+1-\sigma_j(2m_j)-2)}$$

where

$$\Theta_{m_j}(x; x') := \tilde{\psi}(x)\bar{\phi}(x') - \phi(x)\bar{\tilde{\psi}}(x') \quad (4.4)$$

with $\tilde{\psi} := |\phi|^4\phi$. It is obtained from contracting three copies of $|\phi\rangle\langle\phi|$ at the three leaf vertices $\kappa_-(m_j), \kappa(m_j), \kappa_+(m_j)$ which have m_j as their parent vertex.

Now we are ready to begin the induction. Let $\alpha \in \{1, \dots, m_j - 1\}$. Suppose that the kernels $\Theta_{\alpha'}$ have been determined for all $\alpha' > \alpha$. We let $\kappa_-(\alpha), \kappa(\alpha), \kappa_+(\alpha)$ label the three child vertices (of internal or leaf type) of α . Since $\Theta_{\kappa_-(\alpha)}, \Theta_{\kappa(\alpha)}$, and $\Theta_{\kappa_+(\alpha)}$ have already been determined, we can now define

$$\begin{aligned} \Theta_{\alpha}(x; x') &= B_{1;2,3}((U^{(1)}(t_{\alpha} - t_{\kappa_-(\alpha)})\Theta_{\kappa_-(\alpha)}) \otimes (U^{(1)}(t_{\alpha} - t_{\kappa(\alpha)})\Theta_{\kappa(\alpha)}) \otimes (U^{(1)}(t_{\alpha} - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)}))(x; x'). \end{aligned}$$

The induction ends when we obtain the kernel Θ_1 at $\alpha = 1$.

4.4. Key properties of the kernels Θ_{α} . As in [4], we observe that the kernels Θ_{α} satisfy the following properties.

- Θ_{α} can be written as a sum of differences of factorized kernels

$$\Theta_{\alpha}(x; x') = \sum_{\beta_{\alpha}} c_{\beta_{\alpha}}^{\alpha} \chi_{\beta_{\alpha}}^{\alpha}(x) \overline{\psi_{\beta_{\alpha}}^{\alpha}(x')} \quad (4.5)$$

with at most $2^{m_j - \alpha}$ nonzero coefficients $c_{\beta_{\alpha}}^{\alpha} \in \{1, -1\}$.

- The product $\chi_{\beta_{\alpha}}^{\alpha}(x) \overline{\psi_{\beta_{\alpha}}^{\alpha}(x')}$ in (4.5) above is either of the form

$$\begin{aligned} \chi_{\beta_{\alpha}}^{\alpha}(x) \overline{\psi_{\beta_{\alpha}}^{\alpha}(x')} &= (U_{\alpha; \kappa_-(\alpha)} \chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)})(x) \overline{(U_{\alpha; \kappa_-(\alpha)} \psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)})(x')} \\ &\quad A \left[V_{\infty}, (U_{\alpha; \kappa(\alpha)} \chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)})(U_{\alpha; \kappa(\alpha)} \psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}), \right. \\ &\quad \left. (U_{\alpha; \kappa_+(\alpha)} \chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)})(U_{\alpha; \kappa_+(\alpha)} \psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}) \right](x) \end{aligned} \quad (4.6)$$

or

$$\begin{aligned} \chi_{\beta_{\alpha}}^{\alpha}(x) \overline{\psi_{\beta_{\alpha}}^{\alpha}(x')} &= (U_{\alpha; \kappa_-(\alpha)} \chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)})(x) \overline{(U_{\alpha; \kappa_-(\alpha)} \psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)})(x')} \\ &\quad A \left[V_{\infty}, (U_{\alpha; \kappa(\alpha)} \chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)})(U_{\alpha; \kappa(\alpha)} \psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}), \right. \\ &\quad \left. (U_{\alpha; \kappa_+(\alpha)} \chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)})(U_{\alpha; \kappa_+(\alpha)} \psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}) \right](x') \end{aligned} \quad (4.7)$$

for some values of $\beta_{\kappa_-(\alpha)}, \beta_{\kappa(\alpha)}, \beta_{\kappa_+(\alpha)}$ that depend on β_{α} . The trilinear operator A is defines as

$$A[V_{\infty}, f, g](x) := \int \int V_{\infty}(x - y_1, x - y_2) f(y_1) g(y_2) dy_1 dy_2. \quad (4.8)$$

Observe that above, the function $\chi_{\beta_{\alpha}}^{\alpha}$ is either of the quintic form

$$\begin{aligned} \chi_{\beta_{\alpha}}^{\alpha}(x) &= (U_{\alpha; \kappa_-(\alpha)} \chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)})(x) \\ &\quad A \left[V_{\infty}, (U_{\alpha; \kappa(\alpha)} \chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)})(U_{\alpha; \kappa(\alpha)} \psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}), \right. \end{aligned} \quad (4.9)$$

$$\left. (U_{\alpha; \kappa_+(\alpha)} \chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)})(U_{\alpha; \kappa_+(\alpha)} \psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}) \right](x) \quad (4.10)$$

or the linear form

$$\chi_{\beta_{\alpha}}^{\alpha}(x) = (U_{\alpha; \kappa_-(\alpha)} \chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)})(x). \quad (4.11)$$

Accordingly, $\psi_{\beta_\alpha}^\alpha$ respectively is either of linear or quintic form, and the product $\chi_{\beta_\alpha}^\alpha(x)\overline{\psi_{\beta_\alpha}^\alpha}(x')$ always has sextic form (4.6) or (4.7).

- We call the functions $\chi_{\beta_\alpha}^\alpha, \psi_{\beta_\alpha}^\alpha$ in the sum (4.5) *distinguished* if they are a function of $|\phi|^4\phi$. In the product on the right hand side of (4.6), respectively (4.7), at most one of the six factors is distinguished. Indeed, this is true for all regular leaf vertices, and for the distinguished vertex (4.4). By induction along decreasing values of α , it is also true for the internal vertices.

As in [4], we make the following assumption, which simplifies the notation without loss of generality.

Hypothesis 4.1. *We assume that only the functions $\psi_{\beta_1}^1$ and $(\psi_{\beta_{\kappa_+^q(1)}}^{\kappa_+^q(1)})$ are distinguished, where we define*

$$\kappa_+^q(1) := \underbrace{\kappa_+(\kappa_+(\dots(\kappa_+(1))\dots))}_{q \text{ times}}.$$

5. PROOF OF PROPOSITION 2.1

In this section, we prove Proposition 2.1. To simplify notation, we denote the time variable $t_{\ell_j, \alpha}$ by t_α . We denote the subtree of τ_j with root at the vertex α by $\tau_{j, \alpha}$, and let

$$\int \left[\prod_{\alpha' \in \tau_{j, \alpha}} dt_{\alpha'} \right] := \int_{[0, T]^{d_\alpha}} \left[\prod_{\alpha' \in \tau_{j, \alpha}} dt_{\alpha'} \right]$$

be integration with respect to all time variables attached to the internal and root vertices of the subtree $\tau_{j, \alpha}$. Here, the total number of internal and root vertices of the tree $\tau_{j, \alpha}$ is denoted by d_α .

Lemma 5.1. *For $V_\infty \in L^{\frac{1}{1-\epsilon}}(\mathbb{R}^6)$ with small $\epsilon \geq 0$ (or $V_\infty(y, z) = \lambda \delta_0(y) \delta_0(z)$ with $\epsilon = 0$ and $\|V_\infty\|_{L^1} := \lambda$), we have the \dot{H}^{-1} bound*

$$\begin{aligned} & \int \left[\prod_{\alpha' \in \tau_{j, \alpha}} dt_{\alpha'} \right] \|\psi_{\beta_\alpha}^\alpha\|_{\dot{H}^{-1}} \|\chi_{\beta_\alpha}^\alpha\|_{\dot{H}^1} \\ & \leq CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int \left[\prod_{\alpha' \in \tau_{j, \kappa_-(\alpha)}} dt_{\alpha'} \right] \|\psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \int \left[\prod_{\alpha' \in \tau_{j, \kappa(\alpha)}} dt_{\alpha'} \right] \|\psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \int \left[\prod_{\alpha' \in \tau_{j, \kappa_+(\alpha)}} dt_{\alpha'} \right] \|\psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^{-1}} \|\chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1} \end{aligned} \tag{5.1}$$

and the \dot{H}^1 bound

$$\begin{aligned} & \int \left[\prod_{\alpha' \in \tau_{j, \alpha}} dt_{\alpha'} \right] \|\psi_{\beta_\alpha}^\alpha\|_{\dot{H}^1} \|\chi_{\beta_\alpha}^\alpha\|_{\dot{H}^1} \\ & \leq CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int \left[\prod_{\alpha' \in \tau_{j, \kappa_-(\alpha)}} dt_{\alpha'} \right] \|\psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \int \left[\prod_{\alpha' \in \tau_{j, \kappa(\alpha)}} dt_{\alpha'} \right] \|\psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \end{aligned}$$

$$\cdot \int \left[\prod_{\alpha' \in \tau_{j, \kappa_+(\alpha)}} dt_{\alpha'} \right] \|\psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1}. \quad (5.2)$$

Proof. To prove (5.1), we apply the bound (A.3) (or (A.1)) to (4.6) and (4.7) and obtain

$$\begin{aligned} & \int \left[\prod_{\alpha' \in \tau_{j, \alpha}} dt_{\alpha'} \right] \|\psi_{\beta_{\alpha}}^{\alpha}\|_{\dot{H}^{-1}} \|\chi_{\beta_{\alpha}}^{\alpha}\|_{\dot{H}^1} \\ & \leq CT^{3\epsilon} \|V_{\infty}\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0, T)^{d_{\alpha}-1}} \left[\prod_{\alpha' \in \tau_{j, \kappa_-(\alpha)} \cup \tau_{j, \kappa(\alpha)} \cup \tau_{j, \kappa_+(\alpha)}} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \|\psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^{-1}} \\ & = CT^{3\epsilon} \|V_{\infty}\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0, T)^{d_{\kappa_-(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j, \kappa_-(\alpha)}} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \int_{[0, T)^{d_{\kappa_-(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j, \kappa(\alpha)}} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \int_{[0, T)^{d_{\kappa_+(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j, \kappa_+(\alpha)}} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^{-1}}. \end{aligned}$$

In the second step, we performed the t_{α} integral. In the second step, we used the fact that the terms $\psi_{\beta_{\alpha}}^{\alpha}, \chi_{\beta_{\alpha}}^{\alpha}$ depend only on the time variables $t_{\alpha'}$ attached to the vertices of the subtree $\tau_{j, \alpha}$.

To prove (5.2), we apply the bound (A.4) (or (A.2)) to (4.6) and (4.7) and obtain

$$\begin{aligned} & \int \left[\prod_{\alpha' \in \tau_{j, \alpha}} dt_{\alpha'} \right] \|\psi_{\beta_{\alpha}}^{\alpha}\|_{\dot{H}^1} \|\chi_{\beta_{\alpha}}^{\alpha}\|_{\dot{H}^1} \\ & \leq CT^{3\epsilon} \|V_{\infty}\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0, T)^{d_{\alpha}-1}} \left[\prod_{\alpha' \in \tau_{j, \kappa_-(\alpha)} \cup \tau_{j, \kappa(\alpha)} \cup \tau_{j, \kappa_+(\alpha)}} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \|\psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1} \\ & = CT^{3\epsilon} \|V_{\infty}\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0, T)^{d_{\kappa_-(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j, \kappa_-(\alpha)}} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \int_{[0, T)^{d_{\kappa_-(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j, \kappa(\alpha)}} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \\ & \quad \cdot \int_{[0, T)^{d_{\kappa_+(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j, \kappa_+(\alpha)}} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1}. \end{aligned}$$

□

We now recursively apply the bounds in the statement Lemma 5.1 to conclude the proof of uniqueness of solutions to the quintic GP and Hartree hierarchy.

Proposition 5.2. *For the distinguished tree τ_j , we have the bound*

$$\int_{[0, T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \text{Tr} \left(\left| R^{(1, -1)} J_j^1(t, t_1, \dots, t_{m_j}; \sigma_j) \right| \right)$$

$$\leq 2^{m_j} C^{m_j-1} T^{3\epsilon(m_j-1)} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}}^{m_j-1} \|\phi\|_{\dot{H}^1}^{4m_j-3} \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{\dot{H}^{-1}}. \quad (5.3)$$

Proof.

$$\begin{aligned} & \int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \text{Tr} \left(\left| R^{(1,-1)} J_j^1(t, t_1, \dots, t_{m_j}; \sigma_j) \right| \right) \\ &= \int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \text{Tr} \left(\left| R^{(1,-1)} U^{(1)}(t - t_1) \Theta_1 \right| \right) \\ &\leq \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \|\psi_{\beta_1}^1\|_{\dot{H}^{-1}} \|\chi_{\beta_1}^1\|_{\dot{H}^{-1}} \\ &\leq \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \|\psi_{\beta_1}^1\|_{\dot{H}^{-1}} \|\chi_{\beta_1}^1\|_{\dot{H}^1} \\ &\leq \sum_{\beta_{\kappa_-(1)}, \beta_{\kappa(1)}, \beta_{\kappa_+(1)}} CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0,T)^{d_{\kappa_-(\alpha)}}} \left[\prod_{\alpha' \in \tau_j, \kappa_-(\alpha)} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa_-(\alpha)}}^{\kappa_-(\alpha)}\|_{\dot{H}^1} \end{aligned} \quad (5.4)$$

$$\cdot \int_{[0,T)^{d_{\kappa(\alpha)}}} \left[\prod_{\alpha' \in \tau_j, \kappa(\alpha)} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa(\alpha)}}^{\kappa(\alpha)}\|_{\dot{H}^1} \quad (5.5)$$

$$\cdot \int_{[0,T)^{d_{\kappa_+(\alpha)}}} \left[\prod_{\alpha' \in \tau_j, \kappa_+(\alpha)} dt_{\alpha'} \right] \|\chi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^1} \|\psi_{\beta_{\kappa_+(\alpha)}}^{\kappa_+(\alpha)}\|_{\dot{H}^{-1}} \quad (5.6)$$

In the last step, we performed the t_1 integral using (5.1). Now, to bound (5.4) and (5.5), we iterate the H^1 bound (5.2). To bound (5.6), we iterate both (5.1) and (5.2). This establishes (5.3). \square

Proposition 5.3. *For the regular tree τ_j , we have the bound*

$$\begin{aligned} & \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \text{Tr} \left(\left| R^{(1,-1)} J_j^1(t, t_1, \dots, t_{m_j}; \sigma_j) \right| \right) \\ & \leq 2^{m_j} C^{m_j} T^{3\epsilon m_j} \|\phi\|_{\dot{H}^1}^{4m_j+2}. \end{aligned} \quad (5.7)$$

Proof.

$$\begin{aligned} & \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \text{Tr} \left(\left| R^{(1,-1)} J_j^1(t, t_1, \dots, t_{m_j}; \sigma_j) \right| \right) \\ &= \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \text{Tr} \left(\left| R^{(1,-1)} U^{(1)}(t - t_1) \Theta_1 \right| \right) \\ &\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \|\psi_{\beta_1}^1\|_{\dot{H}^{-1}} \|\chi_{\beta_1}^1\|_{\dot{H}^{-1}} \\ &\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \|\psi_{\beta_1}^1\|_{\dot{H}^1} \|\chi_{\beta_1}^1\|_{\dot{H}^1} \end{aligned}$$

From here, we iterate the \dot{H}^1 bound (5.2) to obtain (5.7). \square

Lemma 5.4. *Suppose that $V_\infty \in L^{\frac{1}{1-\epsilon}}$. Then*

$$\|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{\dot{H}^{-1}} \lesssim \begin{cases} \|V_\infty\|_{L^1} \|\phi\|_{\dot{H}^1}^5, & \text{if } \epsilon = 0 \\ \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\phi\|_{\dot{H}^1}^5, & \text{if } \epsilon > 0. \end{cases}$$

Notice that when $\epsilon > 0$, we measure the norm of ϕ in the non-homogeneous Sobolev space H^1 .

Proof. By Strichartz estimates, Sobolev embedding, and Theorem A.1, we have

$$\begin{aligned}
& \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{\dot{H}^{-1}} \\
& \lesssim \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{L^{\frac{6}{5}}} \\
& \leq \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{L^{\frac{3}{2}}} \|\phi\|_{L^6} \\
& \leq \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\phi\|_{L^{\frac{3}{1+3\epsilon}}}^2 \|\phi\|_{L^6} \\
& = \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\phi\|_{L^{\frac{6}{1+3\epsilon}}}^4 \|\phi\|_{L^6} \\
& \lesssim \begin{cases} \|V_\infty\|_{L^1} \|\phi\|_{\dot{H}^1}^5, & \text{if } \epsilon = 0 \\ \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\phi\|_{\dot{H}^1}^5, & \text{if } \epsilon > 0. \end{cases}
\end{aligned}$$

□

We are now ready to conclude the proof of Proposition 2.1.

Proof of Proposition 2.1. Recall from (4.1) that J^k can be decomposed into a product of k one-particle kernels

$$J^k(t, t_1, \dots, t_n; \sigma) = \prod_{j=1}^k J_j^1(t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; \sigma_j),$$

where only one of the factors J_j^1 distinguished. It now follows from Propositions 5.2 and 5.3 that

$$\begin{aligned}
& \int_{[0,T]^{n-1}} dt_1 \cdots dt_{n-1} \text{Tr} \left(\left| R^{(k,-1)} J^k(t, t_1, \dots, t_n; \sigma) \right| \right) \\
& = \int_{[0,T]^{n-1}} dt_1 \cdots dt_{n-1} \prod_{j=1}^k \text{Tr} \left(\left| R^{(1,-1)} J_j^1(t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; \sigma_j) \right| \right) \\
& \leq 2^n C^{n-1} T^{3\epsilon(n-1)} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}}^{n-1} \|\phi\|_{\dot{H}^1}^{4(k+n)-5} \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{\dot{H}^{-1}}.
\end{aligned}$$

Thus, by Lemma 5.4, the difference between two solutions $\gamma := \gamma_1 - \gamma_2$ satisfies

$$\begin{aligned}
& \text{Tr} |R^{(k,-1)} \gamma^{(k)}| \\
& \leq (\#\mathcal{M}_{k,n}) \sup_{\sigma \in \mathcal{M}_{k,n}} \sup_{i=1,2} \int_{[0,T]^n} dt_n \int d\mu_{t_n}^{(i)}(\phi) \text{Tr}(|R^{(k,-1)} J^k(\underline{t}_n; \sigma)|) \\
& \leq \left(CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \right)^{n-1} \int_0^T dt_n \int d\mu_{t_n}^{(i)}(\phi) \|\phi\|_{\dot{H}^1}^{4(k+n)-5} \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{\dot{H}^{-1}} \\
& \leq \begin{cases} \left(C \|V_\infty\|_{L^1} \right)^n \int_0^T dt_n \int d\mu_{t_n}^{(i)}(\phi) \|\phi\|_{\dot{H}^1}^{4(k+n)}, & \text{if } \epsilon = 0 \\ \left(CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \right)^{n-1} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int_0^T dt_n \int d\mu_{t_n}^{(i)}(\phi) \|\phi\|_{\dot{H}^1}^{4(k+n)}, & \text{if } \epsilon > 0 \end{cases} \\
& \leq \begin{cases} \left(C \|V_\infty\|_{L^1} \right)^n TM^{4(k+n)}, & \text{if } \epsilon = 0 \\ \left(CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \right)^{n-1} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} TM^{4(k+n)}, & \text{if } \epsilon > 0 \end{cases} \\
& \rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

for T sufficiently small if $\epsilon > 0$, and for M sufficiently small if $\epsilon = 0$. Thus $\text{Tr}|R^{(k,-1)}\gamma^{(k)}| = 0$. Combining this with the a-priori bound

$$\begin{cases} \text{Tr}|R^{(k,1)}\gamma^{(k)}| < M^{2k}, & \text{if } \epsilon = 0 \\ \text{Tr}|S^{(k,1)}\gamma^{(k)}| < M^{2k}, & \text{if } \epsilon > 0 \end{cases}$$

yields the desired result. Namely,

$$\begin{cases} \text{Tr}|R^{(k,1)}\gamma^{(k)}| = 0, & \text{if } \epsilon = 0 \\ \text{Tr}|S^{(k,1)}\gamma^{(k)}| = 0, & \text{if } \epsilon > 0. \end{cases} \quad \square$$

APPENDIX A. MULTILINEAR ESTIMATES

In this section, we present the key multilinear estimates that we will use to prove our main theorems. For the GP hierarchy, our key estimates are in Proposition A.1. The key estimates for the Hartree hierarchy are in Propositions A.3.

Proposition A.1 (Multilinear estimates for GP).

$$\|(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)(e^{it\Delta}f_4)(e^{it\Delta}f_5)\|_{L_t^1\dot{H}_x^{-1}} \lesssim \|f_1\|_{\dot{H}^{-1}} \prod_{j=2}^5 \|f_j\|_{\dot{H}^1}, \quad (\text{A.1})$$

$$\|(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)(e^{it\Delta}f_4)(e^{it\Delta}f_5)\|_{L_t^1\dot{H}_x^1} \lesssim \prod_{j=1}^5 \|f_j\|_{\dot{H}^1}. \quad (\text{A.2})$$

For the proof, we need

Lemma A.2 (Negative Sobolev norm estimate).

$$\|fg\|_{\dot{H}^{-1}} \lesssim \|f\|_{\dot{W}^{-1,6}} \|g\|_{\dot{W}^{1,\frac{3}{2}}}.$$

Proof. We prove the lemma by the standard duality argument, the product rule and the Sobolev inequality.

$$\begin{aligned} \int fg\bar{h} dx &\leq \|f\|_{\dot{W}^{-1,6}} \|gh\|_{\dot{W}^{1,\frac{6}{5}}} \\ &\lesssim \|f\|_{\dot{W}^{-1,6}} \left(\|g\|_{L^3} \|h\|_{\dot{H}^1} + \|g\|_{\dot{W}^{1,\frac{3}{2}}} \|h\|_{L^6} \right) \\ &\lesssim \|f\|_{\dot{W}^{-1,6}} \|g\|_{\dot{W}^{1,\frac{3}{2}}} \|h\|_{H^1}. \end{aligned}$$

\square

Proof. By Lemma A.2, Sobolev embedding and Strichartz estimates, we prove that

$$\begin{aligned} &\|(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)(e^{it\Delta}f_4)(e^{it\Delta}f_5)\|_{L_t^1\dot{H}_x^{-1}} \\ &\lesssim \|e^{it\Delta}f_1\|_{L_t^2W_x^{-1,6}} \left\| \prod_{j=2}^5 e^{it\Delta}f_j \right\|_{L_t^2\dot{W}_x^{1,\frac{3}{2}}} \\ &\lesssim \|f_1\|_{\dot{H}^{-1}} \left(\|e^{it\Delta}f_2\|_{L_t^2\dot{W}_x^{1,6}} \prod_{j=3}^5 \|e^{it\Delta}f_j\|_{L_t^\infty L_x^6} + \text{three similar terms (by the product rule)} \right) \\ &\lesssim \|f_1\|_{\dot{H}^{-1}} \prod_{j=2}^5 \|f_j\|_{\dot{H}^1} \end{aligned}$$

and

$$\|(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)(e^{it\Delta}f_4)(e^{it\Delta}f_5)\|_{L_t^1\dot{H}_x^1}$$

$$\begin{aligned}
&\lesssim \|e^{it\Delta} f_1\|_{L_t^2 \dot{W}_x^{1,6}} \prod_{j=2}^5 \|e^{it\Delta} f_j\|_{L_t^8 L_x^{12}} + \text{four similar terms (by the product rule)} \\
&\lesssim \|e^{it\Delta} f_1\|_{L_t^2 \dot{W}_x^{1,6}} \prod_{j=2}^5 \|e^{it\Delta} f_j\|_{L_t^8 \dot{W}_x^{1, \frac{12}{5}}} + \text{four similar terms (by the product rule)} \\
&\lesssim \prod_{j=1}^5 \|f_j\|_{\dot{H}^1}.
\end{aligned}$$

□

Recall the definition of the the trilinear operator A in (4.8)

$$A[V_\infty, f, g](x) := \int \int V_\infty(x - y_1, x - y_2) f(y_1) g(y_2) dy_1 dy_2.$$

As an analogue of Proposition A.1, we prove:

Proposition A.3 (Multilinear estimates for Hartree). *Let $\epsilon \geq 0$. Then, we have*

$$\begin{aligned}
&\|A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5)\|_{L_t^1 \dot{H}_x^{-1}} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_m\|_{\dot{H}^{-1}} \prod_{\substack{\ell=1 \\ \ell \neq m}}^5 \|f_\ell\|_{\dot{H}^1}, \quad \forall m = 1, \dots, 5,
\end{aligned} \tag{A.3}$$

and

$$\|A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5)\|_{L_t^1 \dot{H}_x^{-1}} \lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^5 \|f_\ell\|_{\dot{H}^1}. \tag{A.4}$$

We recall the convolution estimates in Beckner [2].

Theorem A.1. *For $1 < p < q < \infty, 1 < s_k < p'/q', k = 1, 2$ and $1/q + 2/p' = \sum 1/s_k, 2 < p'/q'$,*

$$\|A[V_\infty, f, g]\|_{L^q(\mathbb{R}^d)} \leq \|V_\infty\|_{L^p(\mathbb{R}^{2d})} \|f\|_{L^{s_1}(\mathbb{R}^d)} \|g\|_{L^{s_2}(\mathbb{R}^d)}. \tag{A.5}$$

We note that Theorem A.1 also holds for $p = 1$. Indeed, by the change of variables $(x - y, x - z) \rightarrow (y, z)$, Minkowski's inequality, and Hölder's inequality, we have

$$\begin{aligned}
\|A[V_\infty, f, g]\|_{L^q} &= \left\| \int \int V_\infty(y, z) f(x - y) g(x - z) dy dz \right\|_{L_x^q} \\
&\leq \int \int |V_\infty(y, z)| \|f(x - y) g(x - z)\|_{L_x^q} dy dz \\
&\leq \int \int |V_\infty(y, z)| \|f(x - y)\|_{L_x^{s_1}} \|g(x - z)\|_{L_x^{s_2}} dy dz \\
&= \|V_\infty\|_{L^1} \|f\|_{L^{s_1}} \|g\|_{L^{s_2}}.
\end{aligned}$$

Proof of (A.4). For $j \in \{1, 2, 3\}$, we have

$$\begin{aligned}
&\left\| \partial_j \left[A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5) \right] \right\|_{L_t^1 L_x^2} \\
&\leq \|A[V_\infty, (\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5)\|_{L_t^1 L_x^2} \\
&\quad + \text{four similar terms (by the product rule)} \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

By Theorem A.1, Strichartz estimates, and Sobolev embedding,

$$\begin{aligned}
I_1 &\leq \left\| A[V_\infty, (\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \right\|_{L_x^{\frac{12}{5}} \| (e^{it\Delta} f_5) \|_{L_x^{12}}}_{L_t^1} \\
&\lesssim \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \left\| \partial_j e^{it\Delta} f_1 e^{it\Delta} f_2 \right\|_{L_x^{\frac{4}{1+8\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L_x^6} \| (e^{it\Delta} f_5) \|_{L_x^{12}}_{L_t^1} \\
&\leq T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j e^{it\Delta} f_1\|_{L_t^{\frac{2}{1-6\epsilon}} L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^5 \|e^{it\Delta} f_\ell\|_{L_t^8 L_x^{12}} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j e^{it\Delta} f_1\|_{L_t^{\frac{2}{1-6\epsilon}} L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^5 \|e^{it\Delta} f_\ell\|_{L_t^8 \dot{W}_x^{1, \frac{12}{5}}} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^5 \|f_\ell\|_{\dot{H}^1}.
\end{aligned}$$

and similarly for $k \in \{2, 3, 4\}$. For $k = 5$, we have

$$\begin{aligned}
I_5 &\leq \left\| A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \right\|_{L_x^3 \| \partial_j e^{it\Delta} f_5 \|_{L_x^6}}_{L_t^1} \\
&\lesssim \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \left\| e^{it\Delta} f_1 e^{it\Delta} f_2 \right\|_{L_x^{\frac{6}{1+12\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L_x^6} \|\partial_j e^{it\Delta} f_5\|_{L_x^6}_{L_t^1} \\
&\leq T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|e^{it\Delta} f_1\|_{L_t^{\frac{8}{1-24\epsilon}} L_x^{\frac{12}{1+24\epsilon}}} \prod_{\ell=2}^4 \|e^{it\Delta} f_\ell\|_{L_t^8 L_x^{12}} \|\partial_j e^{it\Delta} f_5\|_{L_t^2 L_x^6} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|e^{it\Delta} f_1\|_{L_t^{\frac{8}{1-24\epsilon}} \dot{W}_x^{\frac{12}{5+24\epsilon}}} \prod_{\ell=2}^4 \|e^{it\Delta} f_\ell\|_{L_t^8 \dot{W}_x^{1, \frac{12}{5}}} \|\partial_j e^{it\Delta} f_5\|_{L_t^2 L_x^6} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^5 \|f_\ell\|_{\dot{H}^1}.
\end{aligned}$$

□

Before we proceed to the proof of (A.3), we define $\{P_1, P_2, P_3\}$ to be a conic decomposition of \mathbb{R}^3 . That is, P_j is a Fourier multiplier with symbol $p_j : \mathbb{R}^3 \rightarrow [0, 1]$ such that for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$,

$$\begin{aligned}
p_j(\xi) &= 1 \text{ for } \xi_j^2 \geq 2 \sum_{j' \neq j} \xi_{j'}^2, \\
p_j(\xi) &= 0 \text{ for } \xi_j^2 \leq \frac{1}{2} \sum_{j' \neq j} \xi_{j'}^2, \text{ and} \\
\sum_j p_j(\xi) &= 1 \text{ for all } \xi \in \mathbb{R}^3.
\end{aligned}$$

Observe that $|\xi_j| \sim |\xi|$ on the support of p_j .

Proof of (A.3) when $m = 5$. For $h \in \dot{H}^1(\mathbb{R}^3)$, we have

$$\begin{aligned}
&\int A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)](x) (e^{it\Delta} f_5)(x) \bar{h}(x) dx \\
&= \sum_{j=1}^3 \int \int \int V_\infty(y, z) \partial_j \left[(e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y) (e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) \bar{h}(x) \right]
\end{aligned}$$

$$\begin{aligned}
& \times (\partial_j^{-1} P_j e^{it\Delta} f_5)(x) dy dz dx \\
& = \sum_{j=1}^3 \int \int \int V_\infty(y, z) (\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y) (e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) \\
& \quad \times \bar{h}(x) (\partial_j^{-1} P_j e^{it\Delta} f_5)(x) dy dz dx \\
& \quad + \text{four similar terms (by the product rule)} \\
& =: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

By duality, it now suffices to show that

$$\|I_k\|_{L_t^1} \lesssim T^{3\epsilon} \|V_\infty\|_{L_{\frac{1}{1-\epsilon}}} \|f_5\|_{\dot{H}^{-1}} \left(\prod_{\ell=1}^4 \|f_\ell\|_{\dot{H}^1} \right) \|h\|_{\dot{H}^1} \quad (\text{A.6})$$

holds for $k \in \{1, 2, 3, 4, 5\}$. By Theorem A.1, Strichartz estimates, and Sobolev embedding, we have

$$\begin{aligned}
\|I_1\|_{L_t^1} & \leq \sum_{j=1}^3 \left\| \int \int V_\infty(y, z) (\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y) (e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) dy dz \right\|_{L_x^{\frac{3}{2}}} \\
& \quad \times \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \|_{L_x^6} \|h\|_{L_x^6} \right\|_{L_t^1} \\
& \lesssim \sum_{j=1}^3 \left\| \|V_\infty\|_{L_{\frac{1}{1-\epsilon}}} \|\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2\|_{L_{\frac{3}{1+6\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L^3} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L_x^6} \|h\|_{L_x^6} \right\|_{L_t^1} \\
& \lesssim \sum_{j=1}^3 T^{3\epsilon} \|V_\infty\|_{L_{\frac{1}{1-\epsilon}}} \|\partial_j e^{it\Delta} f_1\|_{L_t^{\frac{2}{1-6\epsilon}} L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^4 \|e^{it\Delta} f_\ell\|_{L_t^\infty L_x^6} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L_t^2 L_x^6} \|h\|_{L_x^6} \\
& \lesssim T^{3\epsilon} \|V_\infty\|_{L_{\frac{1}{1-\epsilon}}} \|f_5\|_{\dot{H}^{-1}} \left(\prod_{\ell=1}^4 \|f_\ell\|_{\dot{H}^1} \right) \|h\|_{\dot{H}^1},
\end{aligned}$$

and similarly (A.6) holds for $k \in \{2, 3, 4\}$. For $k = 5$, we bound $\|I_5\|_{L_t^1}$ by

$$\begin{aligned}
& \sum_{j=1}^3 \left\| \int \int V_\infty(y, z) (e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y) (e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) dy dz \right\|_{L_x^3} \\
& \quad \times \left\| \partial_j^{-1} P_j e^{it\Delta} f_5 \|_{L_x^6} \|\partial_j h\|_{L_x^2} \right\|_{L_t^1} \\
& \lesssim \sum_{j=1}^3 \left\| \|V_\infty\|_{L_{\frac{1}{1-\epsilon}}} \|e^{it\Delta} f_1 e^{it\Delta} f_2\|_{L_{\frac{6}{1+12\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L^6} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L_x^6} \|\partial_j h\|_{L_x^2} \right\|_{L_t^1} \\
& \lesssim \sum_{j=1}^3 \left\| \|V_\infty\|_{L_{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^2 \|e^{it\Delta} f_\ell\|_{L_x^{\frac{12}{1+12\epsilon}}} \prod_{m=3}^4 \|e^{it\Delta} f_m\|_{L_x^{12}} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L_x^6} \|\partial_j h\|_{L_x^2} \right\|_{L_t^1} \\
& \lesssim \sum_{j=1}^3 T^{3\epsilon} \|V_\infty\|_{L_{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^2 \|e^{it\Delta} f_\ell\|_{L_t^{\frac{8}{1-12\epsilon}} \dot{W}_x^{1, \frac{12}{5+12\epsilon}}} \prod_{m=3}^4 \|e^{it\Delta} f_m\|_{L_t^8 \dot{W}_x^{1, \frac{12}{5}}} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L_t^2 L_x^6} \|\partial_j h\|_{L_x^2} \\
& \lesssim T^{3\epsilon} \|V_\infty\|_{L_{\frac{1}{1-\epsilon}}} \|f_5\|_{\dot{H}^{-1}} \left(\prod_{\ell=1}^4 \|f_\ell\|_{\dot{H}^1} \right) \|h\|_{\dot{H}^1}. \quad \square
\end{aligned}$$

Proof of (A.3) when $m \neq 5$. We present the proof for $m = 1$, and note that the proof for $m \in \{2, 3, 4\}$ is similar. i.e. we show that

$$\begin{aligned} & \|A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5)\|_{L_t^1 \dot{H}_x^{-1}} \\ & \lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_1\|_{\dot{H}^{-1}} \prod_{\ell=2}^5 \|f_\ell\|_{\dot{H}^1}. \end{aligned}$$

For $h \in \dot{H}^1(\mathbb{R}^3)$, we have

$$\begin{aligned} & \int A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)](x) (e^{it\Delta} f_5)(x) \bar{h}(x) dx \\ & = \sum_{j=1}^3 \int \int \int V_\infty(y, z) (\partial_j^{-1} P_j e^{it\Delta} f_1)(x-y) \\ & \quad \times \partial_j \left[(e^{it\Delta} f_2)(x-y) (e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) (e^{it\Delta} f_5)(x) \bar{h}(x) \right] dy dz dx \\ & = \sum_{j=1}^3 \int \int \int V_\infty(y, z) (\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2)(x-y) (e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) \\ & \quad \times (e^{it\Delta} f_5)(x) \bar{h}(x) dy dz dx \\ & \quad + \text{four similar terms (by the product rule)} \\ & =: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

By duality, it now suffices to show that

$$\|I_k\|_{L_t^1} \lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_1\|_{\dot{H}^{-1}} \left(\prod_{\ell=2}^5 \|f_\ell\|_{\dot{H}^1} \right) \|h\|_{\dot{H}^1} \quad (\text{A.7})$$

holds for $k \in \{1, 2, 3, 4, 5\}$. By Theorem A.1, Strichartz estimates, and Sobolev embedding, we have

$$\begin{aligned} \|I_1\|_{L_t^1} & \leq \sum_{j=1}^3 \left\| \int \int V_\infty(y, z) (\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2)(x-y) (e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) dy dz \right\|_{L_x^{\frac{3}{2}}} \\ & \quad \times \|e^{it\Delta} f_5\|_{L_x^6} \|h\|_{L_x^6} \Big\|_{L_t^1} \\ & \lesssim \sum_{j=1}^3 \left\| V_\infty \right\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2\|_{L^{\frac{3}{1+6\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L^3} \|e^{it\Delta} f_5\|_{L_x^6} \|h\|_{L_x^6} \Big\|_{L_t^1} \\ & \lesssim \sum_{j=1}^3 T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1\|_{L_t^{\frac{2}{1-3\epsilon}} L_x^{\frac{6}{1+6\epsilon}}} \|\partial_j e^{it\Delta} f_2\|_{L_t^{\frac{2}{1-3\epsilon}} L_x^{\frac{6}{1+6\epsilon}}} \prod_{\ell=3}^5 \|e^{it\Delta} f_\ell\|_{L_t^\infty L^6} \|h\|_{L^6} \\ & \lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_1\|_{\dot{H}^{-1}} \left(\prod_{\ell=2}^5 \|f_\ell\|_{\dot{H}^1} \right) \|h\|_{\dot{H}^1}, \end{aligned}$$

and similarly, (A.7) holds for $k \in \{2, 3, 4\}$. Finally, we bound $\|I_5\|_{L_t^1}$ by

$$\begin{aligned} & \sum_{j=1}^3 \left\| \int \int V_\infty(y, z) (\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot e^{it\Delta} f_2)(x-y) (e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) dy dz \right\|_{L_t^1 L_x^3} \\ & \quad \times \|e^{it\Delta} f_5\|_{L_t^\infty L_x^6} \|\partial_j h\|_{L_x^2} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^3 \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot e^{it\Delta} f_2\|_{L_t^{\frac{3}{2}} L_x^{\frac{9}{2+18\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L_t^3 L_x^9} \|e^{it\Delta} f_5\|_{L_t^\infty L_x^6} \|\partial_j h\|_{L_x^2} \\
&\leq \sum_{j=1}^3 \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1\|_{L_t^2 L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^4 \|e^{it\Delta} f_\ell\|_{L_t^6 L_x^{18}} \|e^{it\Delta} f_5\|_{L_t^\infty L_x^6} \|\partial_j h\|_{L_x^2} \\
&\lesssim \sum_{j=1}^3 T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1\|_{L_t^{\frac{2}{1-6\epsilon}} L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^4 \|e^{it\Delta} f_\ell\|_{L_t^6 \dot{W}_x^{1, \frac{18}{7}}} \|e^{it\Delta} f_5\|_{L_t^\infty \dot{H}_x^1} \|\partial_j h\|_{L_x^2} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_1\|_{\dot{H}^{-1}} \left(\prod_{\ell=2}^5 \|f_\ell\|_{\dot{H}^1} \right) \|h\|_{\dot{H}^1}. \quad \square
\end{aligned}$$

Acknowledgments. The authors would like to express their special appreciation and thanks to their mentors Thomas Chen and Nataša Pavlović for proposing the problem and for various useful discussions. Y.H. would like to thank IHÉS for their hospitality and support while he visited in the summer of 2014. K.T. was supported by NSF grant and DMS-1151414 (CAREER, PI T. Chen)

REFERENCES

- [1] Z. Ammari and F. Nier. Mean field limit for bosons and infinite dimensional phase-space analysis. *Ann. Henri Poincaré*, 9(8):1503–1574, 2008.
- [2] W. Beckner. Multilinear embedding and hardy’s inequality. *Preprint available at arXiv:1311.6747*, 2013.
- [3] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.
- [4] T. Chen, C. Hainzl, N. Pavlović, and R. Seiringer. Unconditional Uniqueness for the cubic Gross-Pitaevskii hierarchy via quantum de finetti. *Comm. Pure Appl. Math. To appear*, 2014.
- [5] T. Chen and N. Pavlović. The quintic NLS as the mean field limit of a boson gas with three-body interactions. *J. Funct. Anal.*, 260(4):959–997, 2011.
- [6] X. Chen and P. Smith. On the unconditional uniqueness of solutions to the infinite radial chern-simons-schrödinger hierarchy. *Preprint available at arXiv:1406.2649*, 2014.
- [7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \mathbb{R}^3 . *Ann. of Math. (2)*, 167(3):767–865, 2008.
- [8] P. Diaconis and D. Freedman. Finite exchangeable sequences. *Ann. Probab.*, 8(4):745–764, 1980.
- [9] L. Erdős, B. Schlein, and H.-T. Yau. Derivation of the Gross-Pitaevskii hierarchy for the dynamics of Bose-Einstein condensate. *Comm. Pure Appl. Math.*, 59(12):1659–1741, 2006.
- [10] L. Erdős, B. Schlein, and H.-T. Yau. Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. *Invent. Math.*, 167(3):515–614, 2007.
- [11] L. Erdős, B. Schlein, and H.-T. Yau. Rigorous derivation of the Gross-Pitaevskii equation with a large interaction potential. *J. Amer. Math. Soc.*, 22(4):1099–1156, 2009.
- [12] L. Erdős, B. Schlein, and H.-T. Yau. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Ann. of Math. (2)*, 172(1):291–370, 2010.
- [13] Z. Han and D. Fang. On the unconditional uniqueness for NLS in \dot{H}^s . *SIAM J. Math. Anal.*, 45(3):1505–1526, 2013.
- [14] K. Hepp. The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.*, 35:265–277, 1974.
- [15] Y. Hong, K. Taliaferro, and Z. Xie. Unconditional uniqueness of the cubic gross-pitaevskii hierarchy with low regularity. *Preprint available at arXiv:1402.5347*, 2014.
- [16] R. L. Hudson and G. R. Moody. Locally normal symmetric states and an analogue of de Finetti’s theorem. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 33(4):343–351, 1975/76.
- [17] T. Kato. On nonlinear Schrödinger equations. II. H^s -solutions and unconditional well-posedness. *J. Anal. Math.*, 67:281–306, 1995.
- [18] K. Kirkpatrick, B. Schlein, and G. Staffilani. Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum dynamics. *Amer. J. Math.*, 133(1):91–130, 2011.
- [19] S. Klainerman and M. Machedon. On the uniqueness of solutions to the Gross-Pitaevskii hierarchy. *Comm. Math. Phys.*, 279(1):169–185, 2008.

- [20] M. Lewin, P. T. Nam, and N. Rougerie. Derivation of Hartree’s theory for generic mean-field Bose systems. *Adv. Math.*, 254:570–621, 2014.
- [21] V. Sohinger. A rigorous derivation of the defocusing cubic nonlinear schrodinger equation on T^3 from the dynamics of many-body quantum systems. *Preprint available at arXiv:1405.3003*, 2014.
- [22] H. Spohn. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Rev. Modern Phys.*, 52(3):569–615, 1980.
- [23] E. Stormer. Symmetric states of infinite tensor products of C^* -algebras. *J. Functional Analysis*, 3:48–68, 1969.

DEPARTMENT OF MATHEMATICS
 THE UNIVERSITY OF TEXAS AT AUSTIN
E-mail address: `yhong@math.utexas.edu`

DEPARTMENT OF MATHEMATICS
 THE UNIVERSITY OF TEXAS AT AUSTIN
E-mail address: `ktaliaferro@math.utexas.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE
 UNIVERSITY OF ILLINOIS AT CHICAGO
E-mail address: `zxie@uic.edu`